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A SHARP L^q-LIOUVILLE THEOREM FOR p-HARMONIC FUNCTIONS

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ABSTRACT

We study L^q-Liouville properties of nonnegative p-superharmonic and, respectively, p-subharmonic functions on a complete Riemannian manifold M. In particular, we prove that every p-harmonic function $u \in L^q(M)$ is constant if $q > p - 1$.

1. Introduction

Liouville-type problems for harmonic functions on noncompact Riemannian manifolds have been extensively studied since the fundamental works of Cheng and Yau [CY], Greene and Wu [GW], and Yau [Y1-2] in the mid-70's. In 1975 Yau [Y1] proved that complete manifolds with nonnegative Ricci curvature have the strong Liouville property, that is, every nonnegative harmonic function on such a manifold is constant. In [CY] Cheng and Yau showed among others that a complete manifold is parabolic if $\liminf_{r\to\infty} V(r)/r^2 < \infty$. Here and in what follows $V(r) = |B(o,r)|$ is the volume of geodesic ball of radius r centered at a fixed point $o \in M$. Recall that M is called **parabolic** if every nonnegative superharmonic function on M is constant or, equivalently, M does not carry a positive Green's function. On the other hand, L^p-Liouville properties for (continuous) nonnegative subharmonic functions were studied e.g. in [GW], [Y2], and in [K2-3]. Greene and Wu [GW] proved that on a complete manifold M, whose sectional curvature is nonnegative outside a compact set, every continuous subharmonic function $u \geq 0$ is either constant or $\int_M u^p = \infty$ for every $p \geq 1$.

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A similar result was obtained by Yau [Y2, Theorem 3 and Appendix] for $p > 1$ without any curvature assumption. More precisely, he showed that on a complete manifold every such u is either constant or $\liminf_{r\to\infty}(1/r)\int_{B(o,r)} u^p > 0$ for every $p > 1$. Karp [K2-3] obtained essentially optimal growth rate for $\int_{B(o,r)} u^p$ by showing that either u is constant or both

$$\liminf_{r\to\infty} \frac{1}{r^2}\int_{B(o,r)} u^p = \infty \quad \text{and} \quad \limsup_{r\to\infty} \frac{1}{r^2 F(r)}\int_{B(o,r)} u^p = \infty$$

hold for every $p > 1$ and every positive nondecreasing function F satisfying $\int_a^\infty dr/rF(r) = \infty$ for some $a > 0$. He also refined the result of Cheng and Yau by showing that M is parabolic if there exists a function F as above, with

$$\limsup_{r\to\infty} \frac{V(r)}{r^2 F(r)} < \infty.$$

There is a vast literature on various Liouville type results and therefore we just refer to the excellent survey articles [G3] by Grigor'yan and [L] by Li for further references and results concerning these and related topics.

Some of the above-mentioned Liouville results have their counterparts for p-harmonic functions as well; see e.g. [H1–4]. For instance, the p-parabolicity of manifolds is studied in [H4] in terms of the growth of $V(r)$. Here a manifold M is called p-**parabolic** if every nonnegative p-superharmonic function on M is constant. On the other hand, several authors have simultaneously extended Yau's strong Liouville result to the p-harmonic case by showing that every nonnegative p-harmonic function on a complete manifold M is constant if a global volume doubling condition and a weak Poincaré inequality hold on M; see [CS], [HR], and [RSV].

In this paper we consider L^q-Liouville properties of p-harmonic functions. A step in this direction was recently taken by Rigoli, Salvatori and Vignati in [RSV]. See also [K1] for earlier related results. Our treatment covers not only p-harmonic functions but also solutions to so-called $\mathcal{A}$-harmonic equations which we introduce next. Let G be an open subset of M and let $TG = \bigcup_{x\in G} T_xM$. Suppose that we are given a map $\mathcal{A}$: $TG \to TG$ such that $\mathcal{A}_x = \mathcal{A} \mid T_xM$: $T_xM \to T_xM$ is continuous for a.e. $x \in G$ and that the map $x \mapsto \mathcal{A}_x(X)$ is a measurable vector field whenever X is. We assume further that there are constants $1 < p < \infty$ and $0 < \alpha \le \beta < \infty$ such that

$$\langle \mathcal{A}_x(h), h\rangle \ge \alpha |h|^p \tag{1.1}$$

and

$$|\mathcal{A}_x(h)| \le \beta |h|^{p-1} \tag{1.2}$$

for a.e. $x \in G$ and for all $h \in T_xM$; in addition, for a.e. $x \in G$

$$\langle \mathcal{A}_x(h) - \mathcal{A}_x(k), h - k \rangle > 0 \tag{1.3}$$

whenever $h \neq k$, and

$$\mathcal{A}_x(\lambda h) = |\lambda|^{p-2} \lambda \mathcal{A}_x(h) \tag{1.4}$$

whenever $\lambda \in \mathbb{R} \setminus \{0\}$.

We say that $\mathcal{A}$ is of type p if it satisfies conditions (1.1)–(1.4) with the constant p. The class of all such $\mathcal{A}$ will be denoted by $\mathcal{A}_p(G)$.

Let $W^{1,p}(G)$ be the Sobolev space of all functions $u \in L^p(G)$ whose distributional gradient ∇u also belongs to $L^p(G)$. We equip $W^{1,p}(G)$ with the norm $\|u\|_{1,p} = \|u\|_p + \|\nabla u\|_p$. The space $W_0^{1,p}(G)$ will be the closure of $C_0^\infty(G)$ in $W^{1,p}(G)$.

A function $u \in W_{\mathrm{loc}}^{1,p}(G)$ is a (weak) solution of the equation

$$-\operatorname{div} \mathcal{A}_x(\nabla u) = 0 \tag{1.5}$$

in G if

$$\int_G \langle \mathcal{A}_x(\nabla u), \nabla \varphi \rangle = 0 \tag{1.6}$$

for all $\varphi \in C_0^\infty(G)$. Continuous solutions of (1.5) are called **$\mathcal{A}$-harmonic** (of type p). It is well-known that every solution of (1.5) has a continuous representative by the fundamental work of Serrin [S]. In the special case $\mathcal{A}_x(h) \equiv |h|^{p-2}h$, $\mathcal{A}$-harmonic functions are called **p-harmonic** and, in particular, if $p = 2$, we obtain harmonic functions.

A function $u \in W_{\mathrm{loc}}^{1,p}(G)$ is a **supersolution** of (1.5) in G if

$$-\operatorname{div} \mathcal{A}_x(\nabla u) \ge 0 \tag{1.7}$$

weakly in G, that is

$$\int_G \langle \mathcal{A}_x(\nabla u), \nabla \varphi \rangle \ge 0 \tag{1.8}$$

for all nonnegative $\varphi \in C_0^\infty(G)$. A function v is called a **subsolution** of (1.5) if $-v$ is a supersolution. It is worth pointing out that the class $C_0^\infty(G)$ of test functions φ in (1.6) and (1.8) can be replaced by $W_0^{1,p}(G)$ if $\nabla u \in L^p(G)$; see [HKM, 3.11].

We present our main theorem (Theorem 2.1) for so-called $\mathcal{A}$-superharmonic and $\mathcal{A}$-subharmonic functions which are closely related to super-and subsolutions of (1.5). These functions form the basis for the nonlinear potential theory of solutions of (1.5) that is developed in [HKM]. A function u: $G \to \mathbb{R} \cup \{\infty\}$ is **$\mathcal{A}$-superharmonic** in G if

(i) u is lower semicontinuous,
(ii) $u \not\equiv \infty$ in each component of G, and
(iii) for each open $D \Subset G$ and each $\mathcal{A}$-harmonic $h \in C(\bar{D})$, the inequality $u \geq h$ on ∂D implies $u \geq h$ in D.

Similarly, a function v is called **$\mathcal{A}$-subharmonic** in G if $-v$ is $\mathcal{A}$-superharmonic in G. Finally, $\mathcal{A}$-superharmonic (resp. $\mathcal{A}$-subharmonic) functions are called **p-superharmonic** (resp. **p-subharmonic**) if $\mathcal{A}_x(h) \equiv |h|^{p-2}h$.

Throughout the paper c will be a positive constant whose actual value may vary even within a line.

2. Main results

In this paper we prove the following L^q-Liouville property for solutions of the $\mathcal{A}$-harmonic equation (1.5). For the formulation of our main result we make a convention $t^0 = 1$ for every $t \in [0, \infty]$. We assume from now on that M is a complete, noncompact Riemannian manifold.

2.1. THEOREM: *Suppose that $1 < p < \infty$, $\mathcal{A} \in \mathcal{A}_p(M)$, $q \in \mathbb{R}$, and $o \in M$. Let u: $M \to [0, \infty]$ be a measurable function and write*

$$v(r) = v_{u,q}(r) = \int_{B(o,r)} u^q .$$

Assume that

$$\int_r^\infty \left(\frac{t}{v(t)} \right)^{1/(p-1)} dt = \infty \tag{2.2}$$

for all $r > 0$. Then u is constant if either

(i) *$q < p - 1$ and u is $\mathcal{A}$-superharmonic in M; or*
(ii) *$q > p - 1$ and u is $\mathcal{A}$-subharmonic in M.*

2.3. Remarks: (1) Observe that (2.2) is required to hold for *all* $r > 0$. This is necessary since a nonnegative, nonconstant $\mathcal{A}$-subharmonic function may vanish identically in a ball $B(o, r_0)$, say, in which case $v(t) = 0$ for $t \leq r_0$ and so (2.2) holds for all $r < r_0$.

(2) The case $q = 0$ is related to the p-parabolicity of M. Indeed, if u is a nonnegative $\mathcal{A}$-superharmonic function in M, then $u^q \equiv 1$ by our convention, and so $v(r) = |B(o,r)|$. The condition (2.2) then implies that M is p-parabolic and hence every nonnegative $\mathcal{A}$-superharmonic function on M is constant; see e.g. [H4] and [H1]. We also remark that the definition of $v_{u,0}(r)$ makes sense for a nonconstant nonnegative $\mathcal{A}$-superharmonic function u regardless of our convention since $u < \infty$ a.e. by e.g. [HKM, 2.10 and 10.2] and, on the other hand, u is positive by [HKM, 7.12].

2.4. COROLLARY:

(i) *If $q < p - 1$ and u is a nonnegative $\mathcal{A}$-superharmonic function in M, with $\int_M u^q < \infty$, then u is constant.*

(ii) *If $q > p - 1$ and u is a nonnegative $\mathcal{A}$-subharmonic function in M, with $\int_M u^q < \infty$, then u is constant. In particular, if u is $\mathcal{A}$-harmonic (not necessarily ≥ 0) in M and $u \in L^q(M)$, with $q > p - 1$, then u is constant.*

Theorem 2.1 is known for sub- and supersolutions of the usual Laplace equation. In fact, Sturm [St] proved 2.1 for solutions of equations $Lu = 0$ associated to Dirichlet forms thus generalizing and further refining the works of Greene and Wu [GW], Yau [Y2], and Karp [K2]. Although the formulation of Theorem 2.1 and the main idea of its proof come from [St], we feel that it will be useful to present this generalization. Furthermore, it is worth pointing out that we prove Theorem 2.1 not only for *sub-* and *supersolutions* of (1.5) but for *$\mathcal{A}$-sub-* and *$\mathcal{A}$-superharmonic* functions as well.

The exponent $q = p - 1$ in Theorem 2.1 is critical in the following sense.

2.5. THEOREM: *Given $1 < p < \infty$ and an integer $n \geq 2$, there exist a complete Riemannian n-manifold M and a nonconstant positive p-harmonic function g in M, with $\int_M g^{p-1} < \infty$.*

In the case $p = n = 2$, such examples of M and g were constructed by Li and Schoen in [LS]. In their example M is the punctured unit disc $\mathbf{B}^2 \setminus \{0\}$ equipped with a suitable conformal change of the standard Euclidean metric and $g(x) = -\log|x|$. It is possible to modify their examples and obtain solutions to 2.5 in the case where $p \geq 2$ is an integer and $n \geq p$. Another example was given by Grigor'yan in [G2]. We choose his approach since we are interested in all possible values of $p \in]1, \infty[$ and $n \geq 2$. It is worth observing that the notion of Green's function for (1.5) will be useful in our construction.

3. L^q-Liouville property

This section is devoted to the proof of Theorem 2.1. We start the proof by collecting some facts from [HKM] in order to create suitable test functions.

3.1. LEMMA: *Let u be a nonnegative nonconstant $\mathcal{A}$-superharmonic function in M, $\kappa \in \mathbb{R}$, $o \in M$, and $R > 0$. For each $k = 1, 2, \ldots$, write $u_k = \min(u, k)$. Then*

(a) *u_k belongs to $W^{1,p}_{\mathrm{loc}}(M)$ and is a supersolution of (1.5);*
(b) *there exists a constant $c > 0$ such that $u_k \geq c$ in $\bar{B}(o, R)$;*
(c) *u_k^κ is bounded in $\bar{B}(o, R)$ and belongs to $W^{1,p}(B(o, R))$;*
(d) *$\varphi_k := u_k^\kappa \psi^p \in W^{1,p}_0(B(o, R))$ if ψ is a nonnegative C^∞ function vanishing identically in $M \setminus B(o, R)$. Furthermore,*

$$\nabla \varphi_k = p u^\kappa \psi^{p-1} \nabla \psi + \kappa \psi_p u^{\kappa - 1} \nabla u\,.$$

Proof: The claim (a) follows from [HKM, 7.2, 7.12]. In order to prove (b), let $c = \inf\{u(x) : x \in \bar{B}(o, R)\}$. Since u is lower semicontinuous and $\bar{B}(o, R)$ is compact, there exists a point $x \in \bar{B}(o, R)$, where $u(x) = c$. By [HKM, 7.12], a nonconstant $\mathcal{A}$-superharmonic function in a domain Ω cannot attain its infimum in Ω. Hence $c > 0$ and (b) holds. The claim (c) now follows from [HKM, 1.18] since $0 < c \leq u_k \leq k$ in $B(o, R)$. Finally, (d) follows from (c) and [HKM, 1.24]. ∎

Similarly, one can prove the following lemma for $\mathcal{A}$-subharmonic functions. We omit the details.

3.2. LEMMA: *Let u be a nonnegative nonconstant $\mathcal{A}$-subharmonic function in M, $\kappa \in \mathbb{R}$, $o \in M$, and $R > 0$. For each $k = 1, 2, \ldots$, write $u_k = \max(u, 1/k)$. Then*

(a) *both u and u_k belong to $W^{1,p}_{\mathrm{loc}}(M)$ and are subsolutions of (1.5);*
(b) *there exists a constant $c < \infty$ such that $1/k \leq u_k \leq c$ in $\bar{B}(o, R)$;*
(c) *u_k^κ is bounded in $\bar{B}(o, R)$ and belongs to $W^{1,p}(B(o, R))$;*
(d) *$\varphi_k := u_k^\kappa \psi^p \in W^{1,p}_0(B(o, R))$ if ψ is a nonnegative C^∞ function vanishing identically in $M \setminus B(o, R)$. Furthermore,*

$$\nabla \varphi_k = p u^\kappa \psi^{p-1} \nabla \psi + \kappa \psi_p u^{\kappa - 1} \nabla u.$$

The proof of 2.1 hinges on the following refinement of a Caccioppoli-type inequality. For positive $\mathcal{A}$-harmonic functions such an inequality was proven in [H2].

3.3. LEMMA: *Fix $o \in M$ and $R > r > 0$. Let $\psi \in C_0^\infty(M)$ such that $0 \le \psi \le 1$, $\psi \equiv 1$ in $\bar{B}(o,r)$, and $\psi \equiv 0$ in $M \setminus B(o,R)$. Let $k \ge 1$ and suppose that either*

(i) *$u \ge 0$ is a nonconstant $\mathcal{A}$-superharmonic function in M, $u_k = \min(u,k)$, and $q < p-1$, $q \ne 0$; or*

(ii) *$u \ge 0$ is a nonconstant $\mathcal{A}$-subharmonic function in M, $u_k = \max(u,1/k)$, and $q > p-1$.*

Then in both cases

(3.4)
$$\int_{B(o,R)} \psi^p |\nabla(u_k^{q/p})|^p \le c \left(\int_{A(r,R)} u_k^q |\nabla\psi|^p \right)^{1/p} \left(\int_{A(r,R)} \psi^p |\nabla(u_k^{q/p})|^p \right)^{(p-1)/p},$$

where $A(r,R) = B(o,R) \setminus \bar{B}(o,r)$ and $c = c(\alpha,\beta,p,q)$.

Proof: Write $\kappa = q-p+1$ and $\varphi_k = u_k^\kappa \psi^p$. In the case (i), $\kappa < 0$ and u_k is a supersolution of (1.5) by Lemma 3.1 (a). Respectively, in the case (ii), $\kappa > 0$ and u_k is a subsolution of (1.5). We can use φ_k as a test function by the condition (d) in Lemma 3.1 (resp. Lemma 3.2). Thus in both cases

$$\kappa \int_{B(o,R)} \langle \mathcal{A}_x(\nabla u_k), \nabla\varphi_k \rangle \le 0,$$

and so

(3.5)
$$-p\kappa \int_{A(r,R)} u_k^\kappa \psi^{p-1} \langle \mathcal{A}_x(\nabla u_k), \nabla\psi \rangle \ge \kappa^2 \int_{B(o,R)} u_k^{q-p} \psi^p \langle \mathcal{A}_x(\nabla u_k), \nabla u_k \rangle.$$

The right hand side has a lower bound,

$$\begin{aligned}\kappa^2 \int_{B(o,R)} u_k^{q-p} \psi^p \langle \mathcal{A}_x(\nabla u_k), \nabla u_k \rangle &\ge \kappa^2 \alpha \int_{B(o,R)} u_k^{q-p} \psi^p |\nabla u_k|^p \\ &\ge \kappa^2 \alpha |p/q|^p \int_{B(o,R)} \psi^p |\nabla(u_k^{q/p})|^p \ge 0,\end{aligned}$$

by (1.1). On the other hand, we use (1.2) and Hölder's inequality to estimate the left hand side from above,

$$\begin{aligned}-p\kappa \int_{A(r,R)} u_k^\kappa \psi^{p-1} \langle \mathcal{A}_x(\nabla u_k), \nabla\psi \rangle &\le p|\kappa|\beta \int_{A(r,R)} u_k^\kappa \psi^{p-1} |\nabla u_k|^{p-1} |\nabla\psi| \\ &\le p|\kappa|\beta \left(\int_{A(r,R)} u_k^q |\nabla\psi|^p \right)^{1/p} \left(\int_{A(r,R)} \psi^p |p/q|^p |\nabla(u_k^{q/p})|^p \right)^{(p-1)/p}.\end{aligned}$$

This proves the lemma. ∎

Proof of Theorem 2.1: The case $q = 0$ is already established in Remarks 2.3. We may thus assume that $q \neq 0$ and $u \geq 0$ is nonconstant. Let $\rho_0 > 0$ and $k_0 \geq 1$ be so large that u_k is nonconstant in $B(o, \rho_0)$ for every $k \geq k_0$. Here u_k is as in Lemma 3.3, that is, u is $\mathcal{A}$-superharmonic and $u_k = \min(u, k)$ if $q < p - 1$, and, respectively, u is $\mathcal{A}$-subharmonic and $u_k = \max(u, 1/k)$ if $q > p - 1$. Fix $R > r > \rho_0$ and let ψ be a cut-off function as in 3.3. Since u_k is nonconstant in $B(o, R)$, all terms in (3.4) are positive. Hence

$$\int_{A(r,R)} u_k^q |\nabla\psi|^p \geq c \frac{\left(\int_{B(o,R)} \psi^p |\nabla(u_k^{q/p})|^p\right)^p}{\left(\int_{A(r,R)} \psi^p |\nabla(u_k^{q/p})|^p\right)^{p-1}}. \tag{3.6}$$

Set

$$v_k(t) = \int_{B(o,t)} u_k^q,\ K_k = \int_{A(r,R)} \psi^p |\nabla(u_k^{q/p})|^p, \text{ and } F_k(t) = \int_{B(o,t)} |\nabla(u_k^{q/p})|^p.$$

Choose ψ such that $|\nabla\psi| \leq 2/(R-r)$. As in [St] we conclude from (3.6) by using properties of ψ that

$$\begin{aligned}
v_k(R) - v_k(r) = \int_{A(r,R)} u_k^q &\geq c(R-r)^p \int_{A(r,R)} u_k^q |\nabla\psi|^p \\
&\geq c(R-r)^p \frac{\left(\int_{B(o,R)} \psi^p |\nabla(u_k^{q/p})|^p\right)^p}{\left(\int_{A(r,R)} \psi^p |\nabla(u_k^{q/p})|^p\right)^{p-1}} \\
&= c(R-r)^p \frac{(F_k(r) + K_k)^p}{K_k^{p-1}} \\
&\geq c(R-r)^p F_k(r) \left(\frac{F_k(r)}{K_k} + 1\right)^{p-1} \\
&= c(R-r)^p F_k(r) \left(\frac{F_k(R)}{F_k(R) - F_k(r)}\right)^{p-1}.
\end{aligned}$$

Hence

$$\left(\frac{(R-r)^p}{v_k(R) - v_k(r)}\right)^{1/(p-1)} \leq c \frac{F_k(R) - F_k(r)}{F_k(R) F_k(r)^{1/(p-1)}}.$$

On the other hand, a simple computation shows that

$$\left(\frac{1}{F_k(r)}\right)^{1/(p-1)} - \left(\frac{1}{F_k(R)}\right)^{1/(p-1)} \geq \min\bigl(1, 1/(p-1)\bigr) \frac{F_k(R) - F_k(r)}{F_k(R) F_k(r)^{1/(p-1)}},$$

and so

$$\left(\frac{1}{F_k(r)}\right)^{1/(p-1)} - \left(\frac{1}{F_k(R)}\right)^{1/(p-1)} \geq c\left(\frac{(R-r)^p}{v_k(R)-v_k(r)}\right)^{1/(p-1)}. \tag{3.7}$$

This holds for every $R > r > \rho_0$. Set $r_i = 2^i r$, $i = 0, 1, \ldots$. Then (3.7) implies that

$$\begin{aligned} F_k(r)^{1/(1-p)} &= \sum_{i=0}^{m-1}\left(F_k(r_i)^{1/(1-p)} - F_k(r_{i+1})^{1/(1-p)}\right) + F_k(r_m)^{1/(1-p)} \\ &\geq c\sum_{i=0}^{m-1}\left(\frac{(r_{i+1}-r_i)^p}{v_k(r_{i+1})-v_k(r_i)}\right)^{1/(p-1)} \geq c\sum_{i=0}^{m-1}\left(\frac{r_{i+1}^p}{v_k(r_{i+1})}\right)^{1/(p-1)} \\ &\geq c\sum_{i=0}^{m-1}\int_{r_{i+1}}^{r_{i+2}}\left(\frac{t}{v_k(t)}\right)^{1/(p-1)} dt = c\int_{2r}^{2^{m+1}r}\left(\frac{t}{v_k(t)}\right)^{1/(p-1)} dt. \end{aligned} \tag{3.8}$$

The rest of the proof can be divided into two parts. Consider first the subcase $0 < q < p-1$ of (i). Recall that now $u \geq 0$ is a nonconstant $\mathcal{A}$-superharmonic function and $u_k = \min(u, k)$. Then $v_k(t) = \int_{B(o,t)} u_k^q \leq \int_{B(o,t)} u^q = v(t)$. This together with the assumption (2.2) and the estimate (3.8) imply that

$$\left(\frac{1}{F_k(r)}\right)^{1/(1-p)} \geq c\int_{2r}^{2^{m+1}r}\left(\frac{t}{v(t)}\right)^{1/(p-1)} dt \to \infty$$

as $m \to \infty$. Hence $F_k(r) = 0$ for every $r > \rho_0$ and $k \geq k_0$, and thus u_k is constant for every $k \geq k_0$. This leads to a contradiction with the assumption that u is nonconstant. Hence the theorem holds for $0 < q < p-1$. The cases $q < 0$ and (ii) can be treated simultaneously. Indeed, $u^q \leq \cdots \leq u_{k+1}^q \leq u_k^q \leq \cdots \leq u_1^q$ in both cases $q < 0$ and (ii). Hence

$$v(t) = \int_{B(o,t)} u^q = \lim_{k\to\infty}\int_{B(o,t)} u_k^q = \lim_{k\to\infty} v_k(t)$$

by the Lebesgue Convergence Theorem. On the other hand,

$$\left(\frac{t}{v_k(t)}\right)^{1/(p-1)} \leq \left(\frac{t}{v_{k+1}(t)}\right)^{1/(p-1)} \leq \left(\frac{t}{v(t)}\right)^{1/(p-1)}.$$

We conclude from the Monotone Convergence Theorem that

$$\int_{2r}^{\infty}\left(\frac{t}{v(t)}\right)^{1/(p-1)} = \lim_{k\to\infty}\int_{2r}^{\infty}\left(\frac{t}{v_k(t)}\right)^{1/(p-1)}. \tag{3.9}$$

Hence

$$F_k(r) \to 0 \tag{3.10}$$

as $k \to \infty$ by (2.2), (3.8), and (3.9). It remains to show that (3.10) forces u to be constant and thus leads to a contradiction. First we observe that Poincaré's inequality states that

$$\int_{B(o,r)} |u_k^{q/p} - a_k|^p \le c \int_{B(o,r)} |\nabla(u_k^{q/p})|^p = cF_k(r)\,, \tag{3.11}$$

where

$$a_k = \fint_{B(o,r)} u_k^{q/p}$$

and c is some constant depending on M, p, o, and r. Next we conclude that

$$a_k \to a := \fint_{B(o,r)} u^{q/p}$$

and, furthermore,

$$\int_{B(o,r)} |u_k^{q/p} - a_k|^p \to \int_{B(o,r)} |u^{q/p} - a|^p$$

as $k \to \infty$. Combining this with (3.10) and (3.11) yields

$$\int_{B(o,r)} |u^{q/p} - a|^p = 0\,.$$

Since this holds for every $r \ge \rho_0$, u is constant, which leads to a contradiction. Hence the theorem is proven. ∎

4. L^{p-1}-integrable p-harmonic function

In this section we construct examples in order to prove Theorem 2.5. We also pose a question on sufficient properties of M that forces nonnegative L^{p-1}-integrable $\mathcal{A}$-superharmonic functions to be constant.

Proof of 2.5: Let $M = \mathbb{R} \times S^{n-1}$ equipped with a metric

$$ds^2 = dt^2 + \varrho^2(t)d\vartheta^2,$$

where $\varrho\colon \mathbb{R} \to]0,\infty[$ is a smooth function and $d\vartheta^2$ is the standard metric of the unit sphere S^{n-1} normalized so that $m_\vartheta(S^{n-1}) = 1$. The manifold M is clearly complete. Let $a(t)$ be the $(n-1)$-measure of $\{t\} \times S^{n-1}$. Thus $a(t) = \varrho^{n-1}(t)$. Then we choose $\varrho(t)$ so that

$$a(t) = \begin{cases} t^{-(1+\varepsilon)}\exp(-t^q), & \text{if } t \geq 1, \\ (-t)^{-(1+\varepsilon)}\exp\big((-t)^q\big), & \text{if } t \leq -1, \end{cases}$$

where $\varepsilon \geq 0$ and

$$q = \frac{p+\varepsilon}{p-1}.$$

We claim that M carries a nonconstant positive p-harmonic function g and, furthermore, $g \in L^{p-1}(M)$ if $\varepsilon > 0$. We construct g so that it depends only on the t-coordinate, i.e. $g = g(t)$, and, furthermore,

$$g(t) \to 0 \quad \text{as } t \to -\infty, \tag{4.1}$$

$$g(t) \to \infty \quad \text{as } t \to \infty, \tag{4.2}$$

and

$$\operatorname{cap}_p\big(\{a\} \times S^{n-1}, \{b\} \times S^{n-1}\big) = \big(g(b) - g(a)\big)^{1-p} \tag{4.3}$$

for all $-\infty < a < b < \infty$. Here the so-called p-capacity

$$\operatorname{cap}_p\big(\{a\} \times S^{n-1}, \{b\} \times S^{n-1}\big)$$

is defined by

$$\operatorname{cap}_p\big(\{a\} \times S^{n-1}, \{b\} \times S^{n-1}\big) = \inf_u \int_M |\nabla u|^p, \tag{4.4}$$

where the infimum is taken over all functions $u \in W^{1,p}_{\text{loc}}(M)$, with $u \equiv 0$ in $]-\infty, a] \times S^{n-1}$ and $u \equiv 1$ in $[b, \infty[\times S^{n-1}$. In particular, (4.1) and (4.3) imply that

$$\begin{aligned} g(t)^{1-p} &= \lim_{a \to -\infty} \operatorname{cap}_p\big(\{a\} \times S^{n-1}, \{t\} \times S^{n-1}\big) \\ &=: \operatorname{cap}_p\big(\{-\infty\} \times S^{n-1}, \{t\} \times S^{n-1}\big). \end{aligned} \tag{4.5}$$

Observe that the limit above exists by basic properties of capacities. Thus g is a sort of Green's function for (1.5) with the pole at "$\{\infty\} \times S^{n-1}$"; see [H1].

By modifying a standard reasoning (cf. e.g. [HKM, 2.11]), we conclude that the limit in (4.5) is given by

$$\operatorname{cap}_p\big(\{-\infty\}\times S^{n-1},\{t\}\times S^{n-1}\big)=\left(\int_{-\infty}^{t}\left(\frac{1}{a(s)}\right)^{1/(p-1)}ds\right)^{1-p},$$

and so

$$g(t)=\int_{-\infty}^{t}\left(\frac{1}{a(s)}\right)^{1/(p-1)}ds. \tag{4.6}$$

On the other hand, it is also easy to see that the function g, defined by (4.6), satisfies conditions (4.1)–(4.3) and that a function v,

$$v(t)=\begin{cases}0, & \text{if } t\le a,\\ \dfrac{g(t)-g(a)}{g(b)-g(a)}, & \text{if } a<t<b,\\ 1, & \text{if } t\ge b,\end{cases}$$

is extremal for (4.4) for every $-\infty<a<b<\infty$. Thus g is p-harmonic in M. Another way to construct g is to write the p-Laplace equation for functions depending only on t and then simply solve the equation; see [G2] for the case $p=2$. We have chosen the approach above in order to emphasize the role of Green's function in this context. To verify that

$$\int_M g^{p-1}=\int_{-\infty}^{\infty}g(t)^{p-1}a(t)\,dt<\infty$$

if $\varepsilon>0$, it is enough to show that

$$\int_{-\infty}^{-1}g(t)^{p-1}a(t)\,dt<\infty \tag{4.7}$$

and

$$\int_{T}^{\infty}g(t)^{p-1}a(t)\,dt<\infty \tag{4.8}$$

for some $T>0$. Suppose that $\varepsilon>0$. Consider first the case $t\le -1$. Recall that

$$a(s)=(-s)^{-(1+\varepsilon)}\exp\big((-s)^q\big),\quad q=\frac{p+\varepsilon}{p-1},$$

for $s \leq -1$. Then

$$\begin{aligned} g(t) &= \int_{-\infty}^{t} \left(\frac{1}{a(s)}\right)^{1/(p-1)} ds = \int_{-\infty}^{t} (-s)^{q-1} \exp\left(\frac{-(-s)^q}{p-1}\right) ds \\ &= \frac{(p-1)^2}{p+\varepsilon} \exp\left(\frac{-(-t)^q}{p-1}\right). \end{aligned}$$

So,

$$\begin{aligned} \int_{-\infty}^{-1} g(t)^{p-1} a(t)\, dt &= \left(\frac{(p-1)^2}{p+\varepsilon}\right)^{p-1} \int_{-\infty}^{-1} \exp\big(-(-t)^q\big)(-t)^{-(1+\varepsilon)} \exp\big((-t)^q\big)\, dt \\ &= \frac{(p-1)^{2(p-1)}}{\varepsilon(p+\varepsilon)^{p-1}} < \infty. \end{aligned}$$

Suppose then that $t \geq 1$. Since

$$a(s) = s^{-(1+\varepsilon)} \exp\big(-s^q\big), \quad q = \frac{p+\varepsilon}{p-1},$$

for $s \geq 1$, we get

$$\begin{aligned} g(t) &= g(1) + \int_{1}^{t} \left(\frac{1}{a(s)}\right)^{1/(p-1)} ds = g(1) + \int_{1}^{t} s^{q-1} \exp\left(\frac{s^q}{p-1}\right) ds \\ &= g(1) + \frac{(p-1)^2}{p+\varepsilon} \left(\exp\left(\frac{t^q}{p-1}\right) - \exp\left(\frac{1}{p-1}\right)\right). \end{aligned}$$

In particular, there exist $T \geq 1$ and c such that

$$g(t)^{p-1} \leq c \exp\big(t^q\big)$$

for all $t \geq T$. Hence

$$\begin{aligned} \int_{T}^{\infty} g(t)^{p-1} a(t)\, dt &\leq c \int_{T}^{\infty} \exp\big(t^q\big) t^{-(1+\varepsilon)} \exp\big(-t^q\big)\, dt \\ &= \frac{c}{\varepsilon T^{\varepsilon}} < \infty. \end{aligned}$$

This proves Theorem 2.5. ∎

Open problem. Here we study the existence of nonconstant, positive, L^{p-1}-integrable $\mathcal{A}$-superharmonic functions in terms of the volume growth. Grigor'yan proved in [G2] that every nonnegative superharmonic function $u \in L^1(M)$ is constant if M is geodesically and stochastically complete. On the other hand, he proved in [G1] that a complete manifold is stochastically complete if

$$\int^{\infty} \frac{r dr}{\log V(r)} = \infty.$$

It is therefore natural to study whether there exists a similar condition in terms of the volume growth that forces every nonnegative $\mathcal{A}$-superharmonic function $u \in L^{p-1}(M)$ to be constant. Unfortunately, we are not able to solve this problem here but we make the following guess.

4.9. CONJECTURE: *Let M be a complete manifold such that*

$$\int^{\infty} \left(\frac{r}{\log V(r)}\right)^{p-1} dr = \infty. \tag{4.10}$$

Then every nonnegative $\mathcal{A}$-superharmonic function $u \in L^{p-1}(M)$ is constant for every $\mathcal{A} \in \mathcal{A}_p(M)$.

We justify the condition (4.10) through the following example. Let M be a spherically symmetric manifold $M = \mathbb{R}^n$ equipped with the metric that is given in polar coordinates $(t, \theta) \in]0, \infty[\times S^{n-1}$ as

$$ds^2 = dt^2 + \psi^2(t) d\theta^2,$$

where $d\theta^2$ is the standard Riemannian metric in S^{n-1} and ψ is a positive smooth function defined in $[0, \infty[$ such that $\psi(0) = 0$ and $\psi'(0) = 1$. Fix $\varepsilon \geq 0$ and choose ψ such that $a(t)$, the $(n-1)$-measure of $\{t\} \times S^{n-1}$, satisfies

$$a(t) = t^{-(1+\varepsilon)} \exp(t^q), \quad \text{with } q = \frac{p+\varepsilon}{p-1},$$

for $t \geq 1$. Write

$$c_0 = \int_1^{\infty} \left(\frac{1}{a(s)}\right)^{1/(p-1)} ds.$$

Then the spherical function

$$g(t, \theta) = \min\left(c_0, \int_t^{\infty} \left(\frac{1}{a(s)}\right)^{1/(p-1)} ds\right)$$

is positive and p-superharmonic in M. In fact, g is p-harmonic in $M \setminus \bar{B}(0,1)$. Observe that $B(0, r) = \{(t, \theta): t < r\}$. In order to study whether M carries any nonconstant, nonnegative, L^{p-1}-integrable p-superharmonic function, it is enough to consider g, the reason for this being the same as in the linear case; see [G2]. For completeness we include the short reasoning. Suppose that u is a nonconstant, nonnegative p-superharmonic function in M. As in Lemma 3.1 we conclude that $\inf\{u(x)\colon x \in \bar{B}(0,1)\} > 0$ and hence $u \geq cg$ in $\bar{B}(0,1)$ for some positive constant c. For each sufficiently large i, we write $g_i = \max(0, g - 1/i)$. Then g_i is p-harmonic in a relatively compact set $D_i := \{g > 1/i\} \setminus \bar{B}(0,1)$. Furthermore, $u \geq cg_i$ in ∂D_i and hence $u \geq cg_i$ in D_i by definition. Letting $i \to \infty$, we conclude that $u \geq cg$ in M. Hence there exists a nonconstant, nonnegative p-superharmonic function $u \in L^{p-1}(M)$ if and only if $g \in L^{p-1}(M)$. Next we distinguish the cases $\varepsilon = 0$ and $\varepsilon > 0$. In both cases

$$\exp(cr^q) \lesssim V(r) \lesssim \exp(r^q),$$

where $0 < c < 1$ and

$$q = \frac{p+\varepsilon}{p-1}.$$

Here $V(r) = |B(0,r)|$ and $A(r) \lesssim B(r)$ means that $A(r) \leq cB(r)$ for some constant c and for sufficiently large $r > 0$. We obtain

$$\int_1^\infty \left(\frac{r}{\log V(r)}\right)^{p-1} dr = \infty \quad \text{and} \quad \int_M g^{p-1} = \infty$$

if $\varepsilon = 0$. On the other hand,

$$\int_1^\infty \left(\frac{r}{\log V(r)}\right)^{p-1} dr < \infty \quad \text{and} \quad \int_M g^{p-1} < \infty$$

if $\varepsilon > 0$. Thus this example gives some indication that 4.9 might be true.

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